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Squeezed States and Nondiagonal P-Representation

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The R-representation and the nondiagonal P-representation for density operator with squeezed-state basis are defined. The special cases for chaotic and laser fields are calculated. The Fokker–Planck equation for the damped harmonic oscillator with squeezed bath is considered and the steady-state solution is given. A special case of the steady-state solution for only a thermal bath is shown. The Pegg–Barnett phase distribution is compared with the radial integration on the generalized P-function.

1. INTRODUCTION

In recent years there has been considerable interest in squeezed states of light, e.g., in quantum optics and in possible applications to gravitational-wave detection (Yuen, 1976; Stoler, 1970a, b; Hollenhorst, 1979; Walls, 1983; Schumaker, 1986; and Special Issues on squeezed states in the *Journal of Modern Optics*, 34(6/7), 1987, and the *Journal of the Optical Society of America B*, 4(10), 1987). Formally, these states are generated from coherent states by an appropriate squeeze operator (Yuen, 1976; Walls, 1983; and Special Issues cited above). In 1976 Yuen mentioned the P-representation with squeezed state basis (see also Adam and Janszky (1990). Recently Wünsche (1996) reintroduced the diagonal quasiprobability distribution functions by using the squeezed-state basis. He used the convolution theorem to express the relation between the coherent-state quasiprobability and squeezed-state quasiprobability (Wünsche, 1996).

Since the introduction of the nondiagonal P-representation (PR) by Drummond and Gardiner (1980) it has found many applications in quantum optics (Drummond and Gardiner, 1980; Drummond *et al.*, 1981; Gilchrist *et al.*, 1997; Zhu and Lu, 1989; Craig and McNeil, 1989; Smith and Gardiner,

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1989; Dörfle and Schenzle, 1986; Sarkar *et al.*, 1986), in contrast to the diagonal P-quasiprobability function introduced by Glauber (1963), which becomes highly singular or negative for quantum states with no classical analogue (Barnett and Knight, 1989; Milburn and Walls, 1983). In this work we consider the nondiagonal PR with squeezed-state basis.

The plan of this paper is as follows. In Section 2 we discuss the quasiprobability functions of the squeezed states. The use of these nonclassical states not only leads to a deeper understanding of the nature of light, but also has potential applicability to quantum detection and communication (Adam and Janszky, 1990; Wünsche, 1996). In Section 3 we introduce the nondiagonal PR for the density operator with squeezed-state basis, and prove the existence theorems for the classes of nondiagonal PR. In Section 4 we discuss some applications, namely the Fokker–Planck equation. In Section 5 we compare the phase distribution obtained from the nondiagonal PR by integrating over the radial variable with the Pegg–Barnett phase distribution. Finally, we draw some conclusions.

2. QUASIPROBABILITY FUNCTIONS

A quasiprobability formulation of quantum mechanics was given first by Wigner (1932), with a characteristic function, associated with the symmetrical order of the annihilation and creation operator, defined by

$$\begin{aligned} C_W(\lambda) &= Tr\{\rho \exp(\lambda a^+ - \lambda^* a)\} \\ &= Tr\{\rho D(\lambda)\} \end{aligned} \tag{2.1}$$

where $D(\alpha)$ is the Glauber displacement operator given by (Glauber, 1963)

$$D(\alpha) = \exp(\alpha a^+ - \alpha^* a), \qquad \text{with} \quad \alpha = |\alpha| e^{i\theta}) \tag{2.2}$$

and a and a^+ are, respectively, the usual annihilation and creation operators of the field.

The Wigner function $W(\alpha)$ is defined as the Fourier transform of the characteristic function $C_W(\lambda)$,

$$W(\alpha) = \frac{1}{\pi^2} \int d^2\lambda \exp(\alpha\lambda^* - \alpha^*\lambda)\, C_W(\lambda) \tag{2.3}$$

The Wigner function has been calculated for the squeezed state (Kim *et al.*, 1989).

The Q-representation is another quasiprobability formulation and is defined as the Fourier transform of the antinormal-ordered characteristic function (Mehta and Sudarshan, 1965)

$$C_A(\lambda) = Tr\{\rho \exp(-\lambda^* a \exp(\lambda a^+)\} \tag{2.4}$$

Alternatively, the Q representation can be defined as

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle \tag{2.5}$$

where $|\alpha\rangle$ is the coherent state defined as $|\alpha\rangle = D(\alpha)|0\rangle$. This function has been discussed for the squeezed state (Kim *et al.*, 1989).

Glauber (1963) and Sudarshan (1963) independently introduced the diagonal PR for the probability density. The PR is defined as the Fourier transform of the normal-ordered characteristic function,

$$C_N(\lambda) = Tr\{\rho \exp(\lambda a^+) \exp(-\lambda^* a)\} \tag{2.6}$$

The three characteristic functions $C_W(\lambda)$, $C_A(\lambda)$, and $C_N(\lambda)$ are related to each other.

The diagonal PR is well defined for classical states, but either it is negative or does not exist for states exhibiting nonclassical behavior, (Barnett and Knight, 1987). As a possible way to avoid the limits of the applicability of the P-representation, Drummond and Gardiner (1980) suggested the so-called positive P-representation (PPR). The PPR is defined over a double-phase space. The PPR is always positive as in the Q-representation. The density operator can be expressed in terms of the PPR with coherent state basis. In what follows we shall introduce the representation using squeezed states as basis.

2.1. R-Representation (RR) for the Density Operator with Squeezed-State Basis

We develop an expansion for any operator in terms of squeezed states.

The squeezed coherent state is defined as (Yuen, 1976; Stoler, 1970a, b; Schumaker, 1986; see also Special Issues cited in Section 1)

$$|\alpha,\xi\rangle = S(\xi) D(\alpha)|0\rangle \tag{2.7}$$

with the Glauber displacement operator $D(\alpha)$ [see (2.1)] and with the squeeze operator $S(\xi)$ given by (Stoler, 1970a, b; Hollenhorst, 1979; Schumaker, 1986)

$$S(\xi) = \exp\left(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi a^{+2}\right) \tag{2.8}$$

where

$$\xi = re^{i\phi}, \qquad 0 \le r < \infty, \qquad 0 \le \phi \le 2\pi \tag{2.9}$$

We shall begin by considering in general a class of operators and then specialize to the case of the density operator later (Adam and Janszky, 1990).

A general quantum operator T may be expressed in terms of its matrix elements connecting states with fixed numbers of quanta as (Glauber, 1963)

$$T = \sum_{n,m}^{\infty} T_{nm}|n\rangle\langle m| \tag{2.10}$$

If we use this expression for T to calculate the matrix element which connects the two squeezed coherent states $\langle\alpha, \xi|$ and $|\beta, \xi\rangle$, we find

$$\langle\alpha, \xi|T|\beta, \xi\rangle = \sum_{n,m}^{\infty} T_{nm}\langle\alpha, \xi|n\rangle\langle m|\beta, \xi\rangle$$

where $\langle n|\alpha, \xi\rangle$ is given by (Yuen, 1976)

$$\langle n|\alpha, \xi\rangle = (n!\cosh r)^{-1/2}\left[\frac{1}{2}\tanh re^{i\phi}\right]^{n/2} \times \exp\left[-\frac{1}{2}|\alpha|^2 + \alpha^2 e^{-i\phi}\frac{\tanh r}{2}\right] H_n[\alpha(e^{i\phi}\sinh 2r)^{-1/2}] \tag{2.11}$$

Thus we get

$$\langle\alpha, \xi|T|\beta, \xi\rangle = T(\alpha, \beta, \xi)(\cosh r)^{-1} \times \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \frac{\tanh r}{2}(\beta^2 e^{-i\phi} + \alpha^{*2}e^{i\phi})\right] \tag{2.12}$$

with

$$T(\alpha, \beta, \xi) = \sum_{n,m} T_{nm}(n!\,m!)^{-1/2} H_n[\beta(\sinh 2re^{i\phi})^{-1/2}] H_m[\alpha^*(\sinh 2re^{-i\phi})^{-1/2}] \times \left[\frac{1}{2}\tanh r\right]^{(n+m)/2} \exp\left[i\frac{\phi}{2}(n-m)\right] \tag{2.13}$$

Due to the overcompleteness of the squeezed coherent states belonging to same squeeze parameter, we have

$$\frac{1}{\pi}\int |\alpha, \xi\rangle\langle\alpha, \xi|\, d^2\alpha = I \tag{2.14}$$

Therefore we may write any quantum operator T in the form

$$T = \frac{1}{\pi^2} \int\int T(\alpha, \beta, \xi)|\alpha, \xi\rangle\langle\beta, \xi|(\cosh r)^{-1} \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 \right] \times \exp\left[\frac{\tanh r}{2} (\beta^2 e^{-i\phi} + \alpha^{*2} e^{i\phi}) \right] d^2\alpha \, d^2\beta \tag{2.15}$$

For the special case when $T = \rho$ (ρ is the density operator), then $T(\alpha, \beta, \xi) = R(\alpha, \beta, \xi)$ (Adam and Janszky, 1990) takes the form of (2.15). As a special case, from (2. 14) and (2.15) when we write $\xi = 0$ we get the results for the coherent states (Glauber, 1963).

3. GENERALIZED P-REPRESENTATION WITH SQUEEZED-STATES BASIS

In this section we develop the Glauber–Sudarshan diagonal PR as an expansion in diagonal squeezed-state projection operators (Yuen, 1976; Adam and Janszky, 1990; Wünsche, 1996)

$$\rho = \int d^2\alpha \, P(\alpha, \alpha^*, \xi)|\alpha, \xi\rangle\langle\alpha, \xi| \tag{3.1}$$

This diagonal representation is considered by Yuen (1976), and some basic properties investigated by Wünsche (1996). Because of the overcompleteness of the squeezed states (Schumaker, 1986; and Special Issues cited above) the diagonal P-function $P(\alpha, \alpha^*, \xi)$ is not unique, and does not always exist as a well-behaved function (Barnett and Knight, 1987).

We introduce a class of generalized PR by expanding in nondiagonal squeezed-state projection operators. Let the density operator have the form

$$\rho = \int P(\alpha, \beta, \xi)\Lambda(\alpha, \beta, \xi) \, d\mu(\alpha, \beta) \tag{3.2}$$

with the projection operator defined as

$$\Lambda(\alpha, \beta, \xi) = \frac{|\alpha, \xi\rangle\langle\beta^*, \xi|}{\langle\beta^*, \xi|\alpha, \xi\rangle} \tag{3.3}$$

By using the definition introduced by Stoler (1970a, b) we get

$$\Lambda(\alpha, \beta, \xi) = S(\xi) \exp(\alpha a^+ - \alpha\beta) \ |0\rangle\langle 0| \exp(\beta a) \, S^+(\xi) \tag{3.4}$$

The projection operator $\Lambda(\alpha, \beta, \xi)$ is analytic in (α, β), and $P(\alpha, \beta, \xi)$ in (3.2) is analogous to the usual P-function. The integration measure $d\mu(\alpha, \beta)$

is left undefined at present: By using various integration measures, a class of generalized PR is generated.

(i) *The diagonal PR.* Let

$$d\mu(\alpha, \beta) = \delta^2(\alpha^* - \beta)\, d^2\alpha\, d^2\beta \tag{3.5}$$

This measure corresponds to the diagonal PR with squeezed-state basis (Yuen, 1976; Adam and Janszky, 1990; Wünsche, 1996).

(ii) *Complex PR*:

$$d\mu(\alpha, \beta) = d\alpha\, d\beta \tag{3.6}$$

(iii) *Positive P-representation*:

$$d\mu(\alpha, \beta) = d^2\alpha\, d^2\beta \tag{3.7}$$

In what follows we prove the following existence theorems.

Theorem 1. A complex P-representation exists for an operator with an expansion in a basis of squeezed number states.

Proof. Let

$$\rho = \sum_{n,m}^{\infty} C_{nm} S(\xi) a^{+m} |0\rangle\langle 0| a^n S^+(\xi)$$

Then, by Cauchy's theorem,

$$\rho = \oint_C \oint_{\bar{C}} P(\alpha, \beta, \xi) \Lambda(\alpha, \beta, \xi)\, d\alpha\, d\beta$$

with

$$P(\alpha, \beta, \xi) = \left(-\frac{1}{4\pi^2}\right) e^{\alpha\beta} \sum_{n,m}^{\infty} C_{nm} n!\, m!\, \alpha^{-m-1} \beta^{-n-1} \tag{3.8}$$

where C, $\bar{C}$ are integration paths enclosing the origin.

Theorem 2. A nondiagonal complex PR with squeezed-state basis exists for an operator with an expansion in number states (Fock states) as

$$\rho = \sum_{n,m=0}^{\infty} C_{nm} |n\rangle\langle m| \tag{3.9}$$

Proof. From (3.9) and Cauchy's theorem we can write

$$\rho = \oint_C \oint_{C'} P(\alpha, \beta, \xi)\Lambda(\alpha, \beta, \xi)\, d\alpha\, d\beta$$

with

$$P(\alpha, \beta, \xi) = \frac{-1}{4\pi^2} \exp\left[-\alpha\beta - (\beta^2 e^{-i\phi} + \alpha^2 e^{i\phi})\frac{\tanh r}{2}\right] \times \sum_{n,m=0}^{\infty} C_{nm}(n!\, m!)^{1/2}(\cosh r)\,(\alpha\beta)^{-1} \times \exp\left[\frac{i\phi}{2}(m-n)\right]\left[\frac{1}{2}\tanh r\right]^{-(n+m)/2} \times \{H_n[\alpha(e^{i\phi}\sinh 2r)^{-1/2}]H_m[\beta(e^{-i\phi}\sinh 2r)^{-1/2}]\}^{-1} \quad (3.10)$$

by using the relation, (2.11).

For $\xi = 0$ the asymptotic forms of $H_n(z)$ for argument z lead to the results of Drummond and Gardiner (1980) with the coherent state. The complex PR given by (3.10) is convergent for $0 \le r < \infty$ and one must be very careful about the roots of $H_n(\alpha)$ and $H_m(\beta)$. By choosing appropriate paths of integration C, C' in the complex phase space of (α, β) the complex PR of (3.10) may be looked at as a weight function.

Theorem 3. If the diagonal PR exists, a corresponding PPR exists, with $P(\alpha, \beta, \xi)$ given by

$$P(\alpha, \beta, \xi) = \frac{1}{4\pi^2}\exp\left(-\frac{|\alpha - \beta^*|^2}{4}\right) \times \left\langle \frac{1}{2}(\alpha + \beta^*), \xi|\rho|\frac{1}{2}(\alpha + \beta^*), \xi\right\rangle \quad (3.11)$$

Proof. Let $P(\acute{\alpha}, \acute{\alpha}^*, \xi)$ be the diagonal PR for the density operator with squeezed state basis (3.1). Then by direct substitution into (3.11) we get

$$P(\alpha, \beta, \xi) = \left(\frac{1}{4\pi^2}\right)\int P(\acute{\alpha}, \acute{\alpha}^*, \xi) \times \exp\left[-\frac{|\alpha - \acute{\alpha}|^2}{2} - \frac{|\beta^* - \acute{\alpha}|^2}{2}\right] d^2\acute{\alpha} \quad (3.12)$$

Next it is necessary to demonstrate that $P(\alpha, \beta, \xi)$ as defined above does represent ρ, so the rhs of equation (3.2) is evaluated,

$$\int\int P(\alpha, \beta, \xi)\Lambda(\alpha, \beta, \xi)\, d^2\alpha\, d^2\beta$$
$$= \left(\frac{1}{4\pi^2}\right)\int\int\int P(\acute{\alpha}, \acute{\alpha}^*, \xi)\Lambda(\alpha, \beta, \xi)$$
$$\times \exp\left[-\frac{|\alpha - \acute{\alpha}|^2}{2} - \frac{|\beta^* - \acute{\alpha}|^2}{2}\right] d^2\acute{\alpha}\, d^2\alpha\, d^2\beta$$

Using the integral in the Appendix, we get

$$\int\int P(\alpha, \beta, \xi)\Lambda(\alpha, \beta, \xi)\, d^2\alpha\, d^2\beta = \int |\acute{\alpha}\rangle\langle\acute{\alpha}| P(\acute{\alpha}, \acute{\alpha}^*, \xi)\, d^2\acute{\alpha} = \rho$$

Theorem 4. The PPR exists for any quantum operator and is given by (3.11).

Proof. In order to show that this represents a quantum density operator in the general case, the characteristic function (2.1) is used. In terms of RR for ρ, (2.15), the characteristic function for normal ordering is

$$C_R(\lambda, \xi) = \frac{1}{\pi \cosh r}\int d^2\alpha\, R(\alpha, \alpha + \lambda, \xi)$$
$$\times \exp\left[-|\alpha|^2 + \frac{\tanh r}{2}(e^{i\phi}\alpha^{*2}) - \lambda^*\alpha\right]$$
$$\times \exp\left[\frac{\tanh r}{2} e^{-i\phi}(\alpha + \lambda)^2\right] \qquad (3.13)$$

by using (A10).

We define the generalized function $P(\alpha, \beta, \xi)$ in the form

$$P(\alpha, \beta, \xi) = \frac{1}{4\pi^4 \cosh r}\int\int d^2\acute{\alpha}\, d^2\acute{\beta}\, R(\acute{\alpha}, \acute{\beta}, \xi)$$
$$\times \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 - |\acute{\alpha}|^2 - |\acute{\beta}|^2\right]$$
$$\times \exp\left[\frac{\tanh r}{2}(\acute{\beta}^2 e^{-i\phi} + \acute{\alpha}^{*2} e^{i\phi}) + \frac{\acute{\alpha}}{2}(\alpha^* + \beta) + \frac{\acute{\beta}}{2}(\alpha + \beta^*)\right] \qquad (3.14)$$

by using RR and equation (3.11). Then the corresponding characteristic function $C_P(\lambda, \xi)$ is

$$C_P(\lambda, \xi) = \frac{1}{4\pi^4 \cosh r} \iiiint d^2\acute{\alpha}\, d^2\acute{\beta}\, d^2\alpha\, d^2\beta\, R(\acute{\alpha}, \acute{\beta}, \xi)$$
$$\times \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 - |\acute{\alpha}|^2 - |\acute{\beta}|^2 \right]$$
$$\times \exp\left[\frac{\tanh r}{2}(\acute{\beta}^2 e^{-i\phi} + \acute{\alpha}^{*2} e^{i\phi}) + \frac{\acute{\alpha}}{2}(\alpha^* + \beta) + \frac{\acute{\beta}}{2}(\alpha + \beta^*) + \lambda^*\beta - \lambda\alpha^* \right]$$

By performing the change of variables α and β to

$$\gamma = \frac{(\alpha + \beta^*)}{2}, \qquad \delta = \frac{(\alpha - \beta^*)}{2} \tag{3.15}$$

we can write the above expression for the characteristic function in the form

$$C_P(\lambda, \xi) = \frac{1}{\pi \cosh r} \int d^2\acute{\alpha}\, R(\acute{\alpha}, \acute{\alpha} + \lambda, \xi)$$
$$\times \exp\left[-|\acute{\alpha}|^2 + \frac{\tanh r}{2}(e^{i\phi}\acute{\alpha}^{*2}) - \lambda^*\acute{\alpha}\right] \tag{3.16}$$
$$\times \exp\left[\frac{\tanh r}{2} e^{-i\phi}(\acute{\alpha} + \lambda)^2 \right]$$

by using (A10) for the integrations with respect to β, δ, and γ. From (3.13) and (3.16) it is clear that the two characteristic functions which have been defined in different ways are the same, and we get the result.

Examples. As an example of a state of the quantized radiation field which possesses a well-behaved PR with coherent state basis we may choose a single-mode chaotic or thermal field (Kral, 1990; Knight and Allen, 1983). Equation (3.11) determines the PPR or any quantum operator, hence we use the chaotic operator to define the PPR for such fields. The density operator for chaotic light is given by (Glauber, 1963)

$$\rho_{ch} = \frac{1}{\pi\langle n \rangle} \int \exp\left(-\frac{|\acute{\alpha}|^2}{\langle n \rangle} |\acute{\alpha}\rangle\langle\acute{\alpha}|\, d^2\acute{\alpha} \right) \tag{3.17}$$

The nondiagonal PPR given by (3.11) has the form

$$P_{ch}(\alpha, \beta, \xi) = \frac{(\cosh r)^{-1}}{4\pi^2 \langle n \rangle \sqrt{K}} \times \exp\left[\frac{1}{K \cosh^2 r} \left(1 + \frac{1}{\langle n \rangle}\right) \left| \frac{\alpha + \beta^*}{2} \right|^2 - \frac{1}{2} (|\alpha|^2 + |\beta|^2) \right] \times \exp\left\{ \frac{\tanh r}{2} \left(1 - \frac{1}{\cosh^2 rK}\right) \times \left[\left(\frac{\alpha + \beta^*}{2}\right)^2 e^{-i\phi} + \left(\frac{\alpha^* + \beta}{2}\right)^2 e^{i\phi} \right] \right\} \tag{3.18}$$

where

$$K = \left(1 + \frac{1}{\langle n \rangle}\right)^2 - (\tanh r)^2 \tag{3.19}$$

Another example of such states is the ideal laser radiation field, which has the PR in the form of a delta function, since the phase of oscillations at high optical frequencies is not usually under control and we have to assume that the phase is uniformly distributed. Thus we may write the density operator for the ideal laser on the form (Perina, 1985)

$$\rho_l = \int \frac{1}{2\pi |\acute{\alpha}|} \delta(|\acute{\alpha}| - \sqrt{\langle n \rangle}) |\acute{\alpha}\rangle\langle\acute{\alpha}| \, d^2\acute{\alpha} \tag{3.20}$$

where $\acute{\alpha} = |\acute{\alpha}| e^{i\psi}$. The nondiagonal PPR for the state of the laser radiation is given by (3.11) by introducing (3.20). The distribution is independent of the phase ψ and therefore it describes a stationary field; in effect we get

$$P_l(\alpha, \beta, \xi) = \frac{1}{8\pi^3 \cosh r} \exp\left\{ -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \frac{\tanh r}{2} \left[\left(\frac{\alpha + \beta^*}{2}\right)^2 e^{-i\phi} + \left(\frac{\alpha^* + \beta}{2}\right)^2 e^{i\phi} \right] \right\} \times \exp\left\{ -\langle n \rangle \left[1 + \frac{\tanh r}{2} (e^{-i\phi} + e^{i\phi}) \right] + \frac{\sqrt{\langle n \rangle}}{2} [(\beta + \beta^*) + (\alpha + \alpha^*)] \right\} \tag{3.21}$$

In the expressions (3.18) and (3.21), when we put $\alpha = \beta^*$ we have the PPR for these fields equal to $(1/N)$ the Q-function for the squeezed states (Walls, 1983; Wünsche, 1996) of these fields, where N is a normalization constant of $P_{ch}(\alpha, \alpha^*, \xi)$. Figures 1 and 2 give $[(1/N\, Q(\alpha)]$ for different values of the squeeze parameter r: (a) $r = 0$ and (b) $r = 1$ (we choose $\phi = 0$ and $\langle n \rangle = 2.5$) for the chaotic light and laser radiation, respectively. The results for the coherent state representation (Mehta and Sudarshan, 1965; Perina, 1985) of chaotic and laser radiation fields can be obtained when we set $\xi = 0$ in (3.18) and (3.21). As Fig. 1 shows, the Gaussian shape of (3.18) for the chaotic field is squeezed as the squeeze parameter r increases. Figure 2 shows that the Gaussian shape of (3.21) for the laser field is squeezed and the origin displaced as the squeeze parameter r increases (Kim *et al.*, 1989). In general we obtain the squeezed-state Q-function (Wünsche, 1996) from our PPR as a special case when we put $\alpha = \beta^*$.

4. FOKKER–PLANCK EQUATION (FPE) FOR A DAMPED HARMONIC OSCILLATOR WITH SQUEEZED BATH

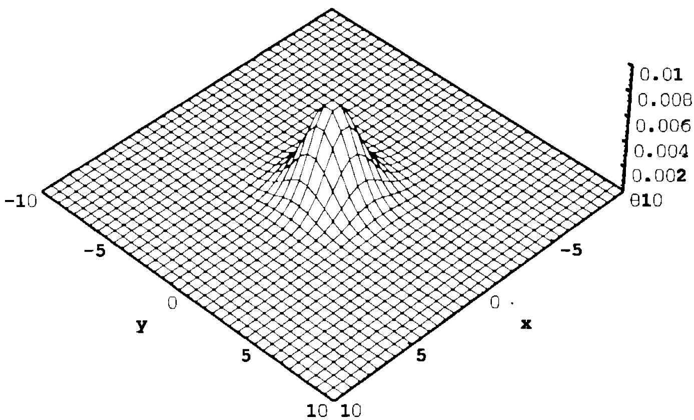
We begin by briefly discussing the standard theory for a damped harmonic oscillator with squeezed bath (Milburn and Walls, 1983; Perina, 1985; Sargent *et al.*, 1974; Louisell, 1973; Walls and Milburn, 1994). Standard treatments of the quantum theory of damping yield the following master equation for the density operator ρ of the harmonic oscillator in the interaction picture (Walls and Milburn, 1994)

$$\frac{\partial \rho}{\partial t} = \frac{\nu}{2}(N+1)\,(2a\rho a^+ - a^+ a\rho - \rho a^+ a) + \frac{\nu}{2} N(2a^+ \rho a - aa^+ \rho - \rho aa^+)$$
$$+ \frac{\nu}{2} M(2a^+ \rho a^+ - a^+ a^+ \rho - \rho a^+ a^+) + \frac{\nu}{2} M^*\,(2a\rho a - aa\rho - \rho aa) \tag{4.1}$$

where ν is the damping constant and N and M are the thermal and squeezed baths, respectivily (Walls and Milburn, 1994). This operator master equation may be converted into an equivalent c-number representation using the Glauber–Sudarshan PR for the density operator (Glauber, 1963; Sudarshan, 1963) in the form (Milburn and Walls, 1983; Walls and Milburn, 1994)

$$\frac{\partial P(\alpha)}{\partial t} = \left\{ \frac{\nu}{2}\left(\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^*}\alpha^* \right) + \frac{\nu}{2}\left(M\frac{\partial^2}{\partial \alpha^2} + M^*\frac{\partial^2}{\partial \alpha^{*2}} \right) + \nu N \frac{\partial^2}{\partial \alpha \alpha^*} \right\} P(\alpha) \tag{4.2}$$

(a)



(b)

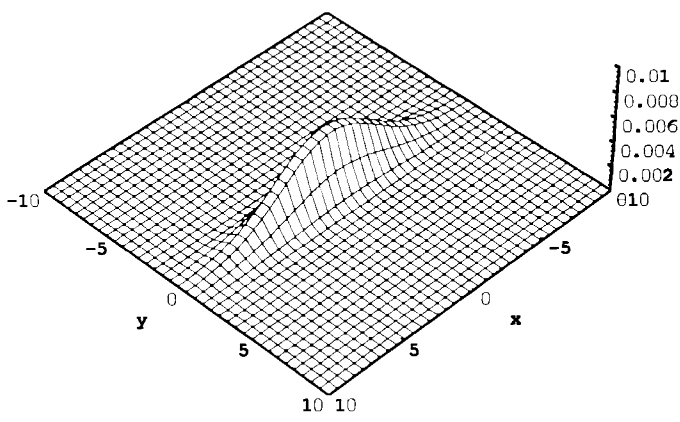


Fig. 1. Positive P-representation for the chaotic radiation field (3.18). Here $\beta = \alpha^*$, $x = Re(\alpha)$, $y = Im(\alpha)$, $\phi = 0$, and $\langle n \rangle = 2.5$. (a) $r = 0$, (b) $r = 1$.

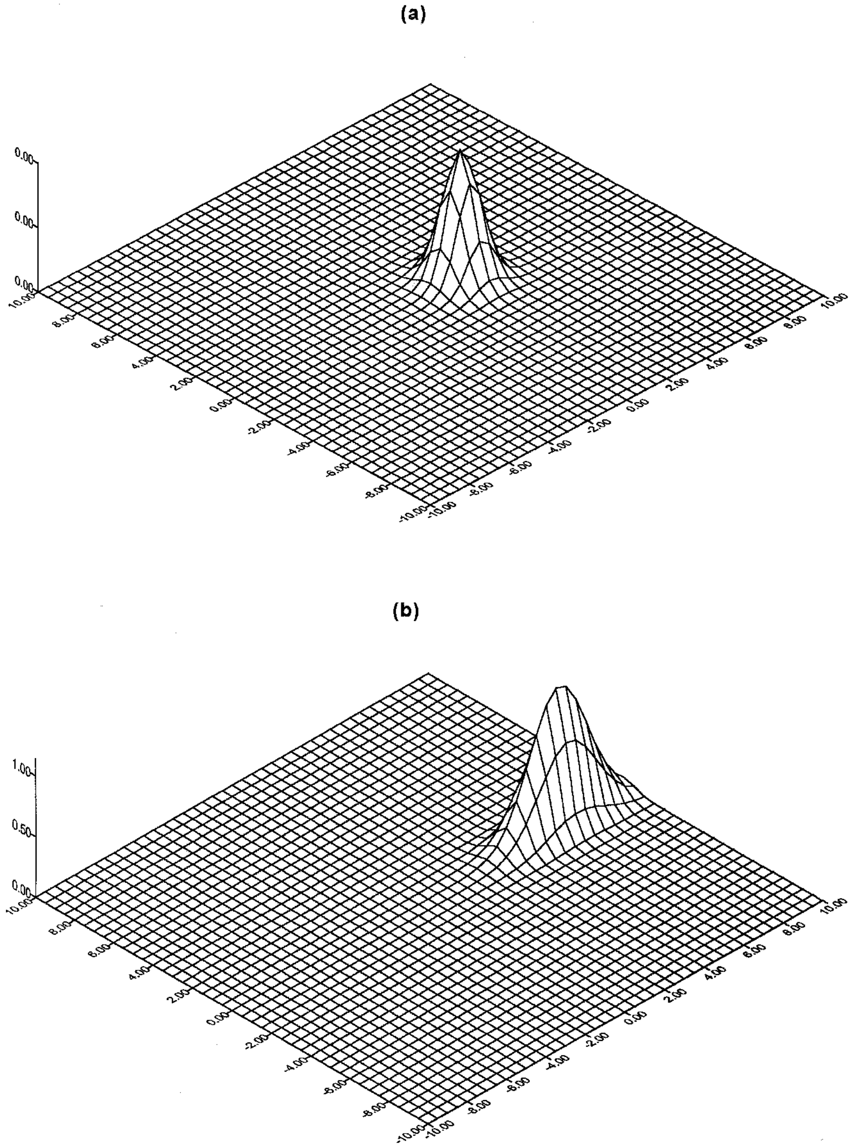


Fig. 2. Positive P-representation for the laser radiation field (3.21). Here $\beta = \alpha^*$, $x = Re(\alpha)$, $y = Im(\alpha)$, $\phi = 0$, and $\langle n \rangle = 2.5$. (a) $r = 0$, (b) $r = 1$.

When $M > N$ this equation has nonpositive-definite diffusion (Walls and Milburn, 1994). While nonsingular representations for $P(\alpha)$ exist for a wide class of states, e.g., thermal states (a Gaussian distribution) and coherent states (a δ-function distribution), a well-behaved positive functions for $P(\alpha)$ does not exist for certain states exhibiting nonclassical behavior (Barnett and Knight, 1987).

The Drummond–Gardiner nondiagonal PR $P(\alpha)$ comes from the expansion of the nondiagonal projection operator of ρ. For the given master equation, the corresponding FPE for $P(\alpha, \beta)$ has been derived by many authors (Drummond and Gardiner, 1980; Drummond *et al.*, 1981; Gilchrist *et al.*, 1997; Zhu and Lu, 1989; Smith and Gardiner, 1989; Dorfle and Schenzle, 1986; Sarkar *et al.*, 1986; McNeil and Craig, 1990; Lu *et al.*, 1989; Walls and Milburn 1994) for different models.

We proceed by substituting (3.2) into the equation of motion for the density operator (4.2) and then use standard techniques (Drummond *et al.*, 1981) for deriving the FPE for the density operator with squeezed-state expansion.

4.1. Operator Identities

We give some operator identities which will be used to derive the FPE. By using the projection operator given by (3.4) and the techniques of (Drummond and Gardiner, 1980; Drummond *et al.*, 1981; Gilchrist *et al.*, 1997), we obtain

$$a\Lambda(\alpha, \beta, \xi) = \left[a\cosh r - e^{i\phi}\left(\beta + \frac{\partial}{\partial\alpha}\right)\sinh r \right]\Lambda\left(\alpha, \beta, \xi\right) \tag{4.3a}$$

$$a^{+}\Lambda(\alpha, \beta, \xi) = \left[\left(\frac{\partial}{\partial\alpha} + \beta\right)\cosh r - e^{-i\phi}\alpha\sinh r \right]\Lambda\left(\alpha, \beta, \xi\right) \tag{4.3b}$$

$$\Lambda(\alpha, \beta, \xi)a^{+} = \left[\beta\cosh r - e^{-i\phi}\left(\frac{\partial}{\partial\beta} + \alpha\right)\sinh r \right]\Lambda\left(\alpha, \beta, \xi\right) \tag{4.3c}$$

$$\Lambda(\alpha, \beta, \xi)a = \left[\left(\frac{\partial}{\partial\beta} + \alpha\right)\cosh r - e^{i\phi}\beta\sinh r \right]\Lambda(\alpha, \beta, \xi) \tag{4.3d}$$

By using these operator identities, and integration by parts, we can cast (4.1) in the form

$$\frac{\partial P(\alpha, \beta, \xi)}{\partial t} = \frac{\nu}{2}\left\{\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\beta}\beta\right) + D_{11}\frac{\partial^2}{\partial\alpha^2} + 2D_{12}\frac{\partial^2}{\partial\alpha\partial\beta} + D_{22}\frac{\partial^2}{\partial\beta^2}\right\}P(\alpha, \beta, \xi) \tag{4.4a}$$

where

$$D_{11} = [M^* \sinh^2 re^{2i\phi} + M \cosh^2 r + N \sinh 2re^{i\phi} \tag{4.4b}$$

$$D_{12} = D_{21} = \frac{1}{2}\left[M^* \sinh 2re^{i\phi} + M \sinh 2re^{-i\phi} + 2N(2 \sinh^2 r + 1) \right] \tag{4.4c}$$

$$D_{22} = [M \sinh^2 re^{-i\phi} + M^* \cosh^2 r + N \sinh 2re^{-i\phi} \tag{4.4d}$$

This differs from the corresponding equation of motion for the P-function only through the squeeze-dependent diffusion coefficients, which are given by (4.4) rather than in (4.2). This is sufficient to give a positive-definite diffusion matrix when the bath is in a squeezed state. The solution for the above equation may be found exactly (Milburn and Walls, 1983; McNeil and Craig, 1990; Lu *et al.*, 1989; Walls and Milburn, 1994).

The steady-state solution is given by

$$P_{ss,\,M}(\alpha, \beta, \xi) = N_1 \exp\left[\frac{\nu}{4(D_{21} - D_{11}\, D_{22})} (D_{11}\, \alpha^2 - 4D_{12}\, \alpha\beta + D_{22}\beta^2) \right] \tag{4.5}$$

where N_1 is the normalization condition, and D_{11}, D_{12}, and D_{22} are given by (4.4).

In the case of a thermal bath, $M = 0$, and $N = (e^{\hbar\omega/kT} - 1)^{-1}$ the thermal photon number, with T the temperature of the reservoir, then the steady-state solution has the form

$$P_{ss}\,(\alpha, \beta, \xi) = \frac{A_0}{2\sqrt{\pi}} \exp\left[-\left(\frac{K}{2}\right)(b\alpha^2 - 2c\alpha\beta + b^*\, \beta^2) \right] \tag{4.6}$$

with $K = 1/(|b|^2 - c^2)$ and b, c given by

$$b = \left(N + \frac{1}{2}\right) \sinh 2re^{i\phi}, \qquad c = (2N + 1) \sinh^2 r + N$$

Here A_0 is a normalization constant, which depends on the integration contours C and C'. Notice that when there is no squeezing (i.e., $\xi = 0$), one can set $\beta = \alpha^*$ in equation (4.6), which reduces to the Glauber–Sudarshan P-distribution of chaotic radiation fields (Glauber, 1963) and the integration in this case is over the whole α plane. With this we conclude this section. In the next section we turn our attention to the phase distribution function.

5. PHASE DISTRIBUTION

The phase properties of the field have attracted considerable interest (Susskind and Glogower, 1964; Barnett and Pegg, 1989, 1990; Ban, 1991, 1993; Vaccaro and Pegg, 1989; Chizhov *et al.*, 1993; Garraway and Knight, 1992, 1993; Luks and Perinova, 1994; Leonhardt and Jex, 1994; Tanas *et al.*, 1993; also see Special Issue of *Physica Scripta*, **T48**, 1993). In the Pegg–Barnett phase operator (Barnett and Pegg, 1989, 1990) formalism, all physical quantities are calculated in $(s + 1)$–dimensional space. After all calculations are completed, s is made infinite. The Pegg–Barnett phase operator and its eignstates are defined in $(s + 1)$-dimensional space Ψ spanned by the number states $|0\rangle$, $|1\rangle$, . . ., $|s\rangle$. The Hermitian phase operator is defined as

$$\Phi_{PB} = \sum_{m=0}^{s} |\eta_m\rangle \, \eta_m \, \langle \eta_m| \tag{5.1}$$

A complete orthonormal basis of $(s + 1)$ phase states is defined on Ψ as

$$|\eta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(-in\eta_m) \, |n\rangle \tag{5.2}$$

which are eigenstates of the operator (5.1) with eigenvalues η_m, where

$$\eta_m = \eta_0 + \frac{2\pi m}{s+1} \qquad (m = 0, 1, 2, \ldots, s) \tag{5.3}$$

Physical states are expressed as $|\psi\rangle = \Sigma_{m=0}^{s} \alpha_m |m>$. Thus, in the Pegg–Barnett formalism, the average value of a phase quantity can be calculated by

$$\langle\psi|\Phi_{PB}|\psi\rangle = \sum_{m=0}^{s} \eta_m |\langle\eta_m|\psi\rangle|^2 \tag{5.4}$$

where $|\langle\eta_m|\psi\rangle|^2$ gives the probability of being in the phase state $|\eta_m\rangle$. We take the limit as $s \to \infty$, to have

$$\langle\psi|\Phi_{PB}|\psi\rangle = \int_{-\pi}^{\pi} \eta P^{PB}(\eta) \, d\eta \tag{5.5}$$

where the continuum phase distribution $P^{PB}(\eta)$ is introduced by

$$P^{PB}(\eta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\eta_m|\psi\rangle|^2 \tag{5.6}$$

The distribution (5.6) is normalized such that

$$\int_{-\pi}^{\pi} P^{PB}(\eta) \, d\eta = 1 \tag{5.7}$$

Once the phase distribution function P^{PB} (η) is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner by integration over η.

The phase distribution properties for the squeezed states are discussed in Ban (1991, 1993). The phase distribution is compared to the phase distribution obtained from the generalized P-representation by integrating over the radii (Chizhov *et al.*, 1993; Garraway and Knight, 1992, 1993; Luks and Perinova, 1994; Leonhart and Jex, 1994; Tanas *et al.*, 1993; and *Physica Scripta*, **T48**, 1993).

From (2.11) and (5.2), we get

$$|\langle\eta_m|\alpha, \xi\rangle|^2 = \frac{1}{s+1} \sum_{n,k=0}^{s} (\langle n|\alpha, \xi\rangle)\, (\langle k|\alpha, \xi\rangle)^* e^{-i(n-k)\eta_m} \tag{5.8}$$

Then the distribution function takes the form

$$P^{PB}(\eta) = \frac{1}{2\pi} \sum_{n,k=0}^{\infty} (\langle n|\alpha, \xi\rangle)(\langle k|\alpha, \xi\rangle)^* e^{-i(n-k)\eta} \tag{5.9a}$$

$$P^{PB}(\theta) = \frac{1}{2\pi} \left\{1 + 2\, Re \left(\sum_{n,k=0,n>k}^{\infty} \rho_{nk} e^{-i(n-k)\theta} \right)\right\} \tag{5.9b}$$

This is the formula for the Pegg–Barnett formalism.

Another phase distribution can be obtained by integrating the nondiagonal positive P-function over the radial variables $|\alpha|$ and $|\beta|$ (Ban, 1991, 1993; Vaccaro and Pegg, 1989; Chizhov *et al.*, 1993; Garraway and Knight, 1992, 1993; Luks and Perinova, 1994). When $\alpha^* = \beta$, we have the special case

$$P_{PP}(\theta) = 4\pi \int_0^\infty P(\alpha, \alpha^*)|\alpha|\, d|\alpha| \tag{5.10}$$

An illustration of the two types of phase distribution can be made using a coherent state $|\alpha\rangle$ (Glauber, 1963) as the quantum state of interest. The Pegg–Barnett phase distribution has to be found by a numerical summation of (5.9). However, to derive the positive P-representation (Wünsche, 1996; Drummond and Gardiner, 1980; Drummond *et al.*, 1981; Gilchrist *et al.*, 1997) phase distribution we obtain [using the special case of (3.11) when $\xi = 0$, $\alpha = \beta^*$, and ρ given by (3.9)],

$$P_{PP}(\theta) = \frac{1}{2\pi} \left\{1 + 2\, Re \left(\sum_{n,m=0,n>m}^{\infty} \rho_{nm} \frac{\Gamma[(n+m)/2+1]}{(n!\, m!)^{1/2}} e^{-i(n-m)\theta} \right)\right\} \tag{5.11}$$

where the coefficients $\Gamma[(n + m)/2 + 1]/\, (n!\, m!)^{1/2}$ distinguish between the two distributions. The PPR (3.11) can be written for the density operator with number state expansion (3.9) as (for real ξ)

$$P(\alpha, \alpha^*, \xi) = \frac{1}{4\pi^2} \sum_{n,m=0}^{\infty} \frac{\rho_{nm}(\frac{1}{2}\tanh r)^{(n+m)/2}}{(n!\, m!)^{1/2} \cosh r} \times \exp\left[-|\alpha|^2 - (\alpha^{*2} + \alpha^2)\frac{\tanh r}{2}\right] \times H_n\left[\frac{|\alpha|(e^{i\theta} + e^{-i\theta}\tanh r)}{(2\tanh r)^{1/2}}\right] H_m\left[\frac{|\alpha|(e^{-i\theta} + e^{i\theta}\tanh r)}{(2\tanh r)^{1/2}}\right] \tag{5.12}$$

where ρ_{nm} is the Fock density matrix element. Then we may write the phase distribution [by using (5.12) in (5.10), Hermite polynomial expansion (Kim *et al.*, 1989), and integration (Gradshteyn and Ryzhik, 1980)]

$$P_{PP}(\theta) = \frac{1}{2\pi}\left\{1 + 2\,Re\left(\sum_{n,m=0}^{\infty} \rho_{nm} G(n, m, \theta) \exp[-i(n - m)\theta]\right)\right\} \tag{5.13}$$

where

$$G(n, m, \theta) = \sum_{k=0}^{n/2} \sum_{s=0}^{m/2} \frac{(\frac{1}{2}\tanh r)^{k+s} (n!\, m!)^{1/2} (-1)^{k+s}}{k!\, s!\, (n - 2k)!\, (m - 2s)!\, \cosh r} \times \Gamma\left[\frac{(k + n)}{2} - m - s + 1\right] \times \Delta^{k+s-1-(n+m)/2} (1 + e^{-2i\theta}\tanh r)^{m-2s} (1 + e^{2i\theta}\tanh r)^{n-2k} \times \exp[i(2k - 2s)\theta] \tag{5.14a}$$

and

$$\rho_{nm} = (\langle n|\alpha_0, \xi_0\rangle)(\langle m|\alpha_0, \xi_0\rangle)^*, \qquad \Delta = 1 - \tanh r \cos(2\theta) \tag{5.14b}$$

The formula (5.14) allows for calculations of the phase distribution for any squeezed state with known ρ_{nm} (matrix elements) and comparison to the Pegg–Barnett phase distribution. Formula (5.14) generalizes the results obtained from the s-parametrized quasiprobability distributions given in Tanas *et al.* (1993; also see Special Issue of *Physica Scripta*, **T48**; 1993). The positive P-function may be written in the form (ξ is real)

$$P(\alpha, \alpha^*, \xi) = \frac{1}{4\pi^2} \langle \alpha, r|\rho|\alpha, r\rangle \tag{5.15}$$

with the density operator given by (for α_0 and r_0 real constants)

$$\rho = |\alpha_0, r_0\rangle\langle\alpha_0, r_0| \tag{5.16}$$

Then we get

$$P(\alpha, \alpha^*, \xi) = \frac{1}{4\pi} g \exp[-A|\alpha|^2 - 2B|\alpha|]$$

where

$$\begin{aligned} g &= \frac{1}{\pi \cosh(r_0 - r)} \exp\{-|\alpha_0|[(\cosh 2r_0 + \sinh 2r_0 \\ &\quad \times - \tanh(r_0 - r))(\cosh r_0 + \sinh r_0)^2]\} \\ A &= \cosh 2r + \sinh 2r \tanh(r_0 - r) \\ &\quad + [\cosh 2r \tanh(r_0 - r) + \sinh 2r] \cos 2\theta \\ B &= -\alpha_0 \left(1 + \frac{\sinh(r_0 + r)}{\cosh(r_0 - r)}\right) \cos\theta \end{aligned}$$

By using (5.10) we get

$$P_{PP}(\theta) = g\left\{\frac{1}{2A} - \frac{B}{2A}\left(\frac{\pi}{A}\right)^{1/2} \exp\left(\frac{B^2}{A}\right)\left[1 - \mathrm{Erf}\left(\frac{B}{A^{1/2}}\right)\right]\right\} \tag{5.17}$$

Formula (5.17) is valid for both small and large α_0. For $r = 0$ we have the result for the squeezed state (Tanas *et al.*, 1993; see also Special Issue of *Physica Scripta*, **T48**, 1993) on the coherent basis. In the phase distribution calculations we use $|\beta, \xi\rangle = D(\beta_0)S(\xi)|0\rangle$, the identical definition of the squeezed state. Figures 3 and 4 show plots of the Pegg–Barnett phase distribution $P^{PB}(\theta)$ and the phase produced from integrating the positive P-representation over radii in one complex plane $P_{PP}(\theta)$ in the case $\xi = 0$. Both figures show the constant distribution for $\alpha = 0$ and how the peak around $\theta = 0$ starts to build up as $|\alpha|$ increases. The plot in Fig. 4 is similar to the phase distribution obtained from integrating the Q-function over radius (Garraway and Knight, 1992, 1993; also see Special Issue of *Physica Scripta*, **T48**, 1995). Figure 5 shows the three-dimensional picture of the phase distribution (5.17) with increasing squeeze r. We note the bifurcation (Special Issue, *Physica Scripta*, **T48**, 1993). However, we remark here that the direction of the bifurcation may be switched according to the sign of the parameter r_0.

6. CONCLUSIONS

General classes of quasiprobabilities, i.e., Wigner, Q-function, and P-representation, have been discussed briefly. We introduced the R-representa-

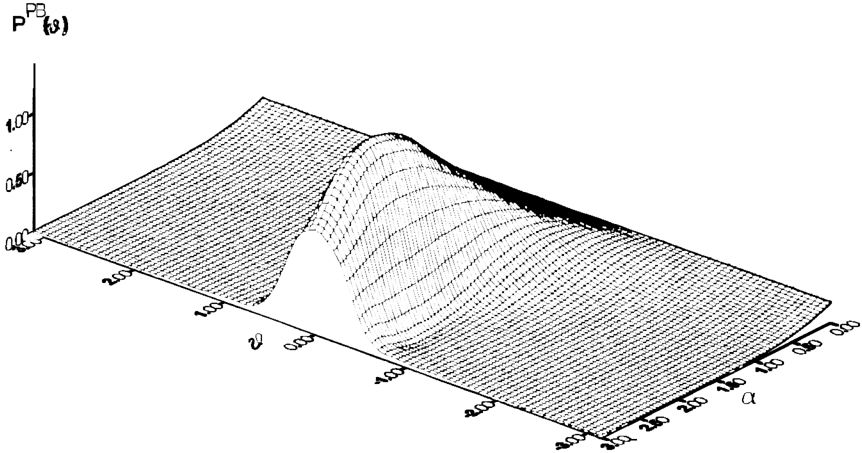


Fig. 3. The phase distribution $P^{PB}(\theta)$, equation (5.9) (i.e., for $r = 0$). Here θ: $\pi \to -\pi$ and α: $0 \to 3$.

tion for the density operator using the squeezed state as basis, which generalizes Glauber (1963) and Adam and Janszky (1990). The nondiagonal P-representation with squeezed-state basis was defined for classes of complex and positive representations, which generalizes the work of Drummond and Gardiner (1980). The new representation for the density operator generalizes the work of Wünsche (1996) since the measure in (3.5) gives us the diagonal PR. But the positive PR produces the Q-function in that work. We found explicit expressions for the positive P-representation for the two natural fields, which are extensions to results given in Perina (1985). We introduced an application of the genuine representation to the Fokker–Planck equation for

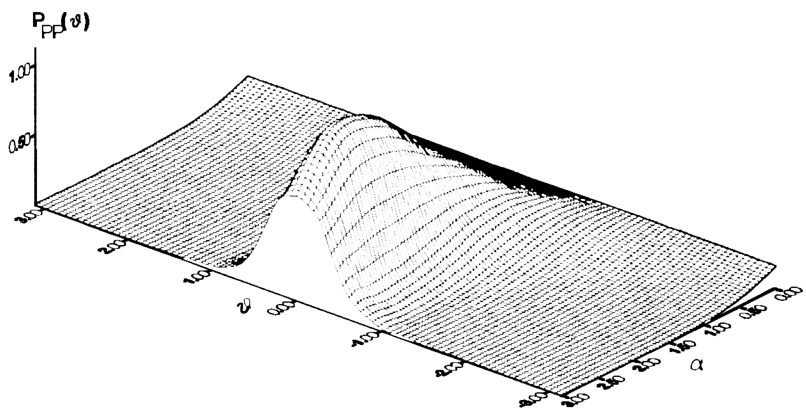
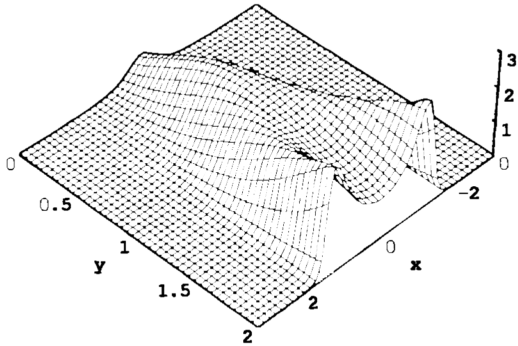


Fig. 4. The phase distribution $P_{PP}(\theta)$, equation (5.17) (i.e., for $r_0 = r = 0$). Here θ: $\pi \to -\pi$ and α: $0 \to 3$.

(a)



(b)

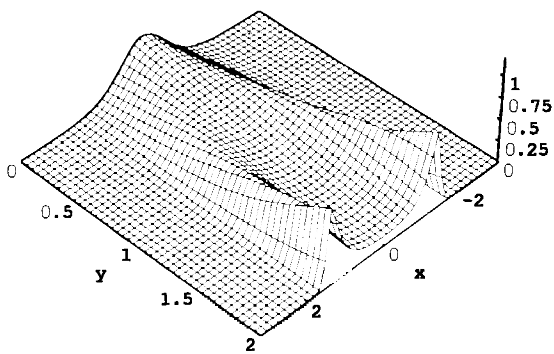
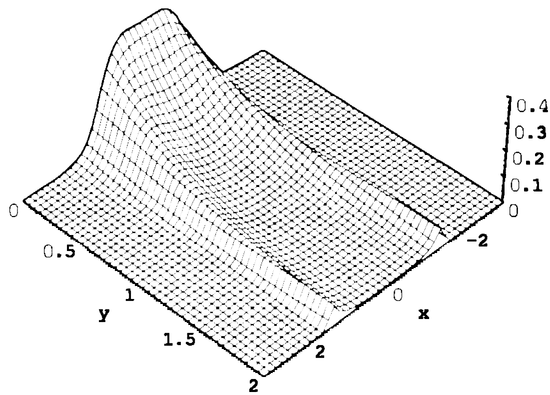


Fig. 5. The phase distribution $P_{PP}(\theta)$, equation (5.17), against the initial squeeze parameter r_0; $\alpha_0 = 1$, and r takes the following values: (a) $r = -0.5$, (b) $r = 0$, (c) $r = 0.5$, and (d) $r = 1$. Here $x = \theta$ and $y = r_0$.

{c}



{d}

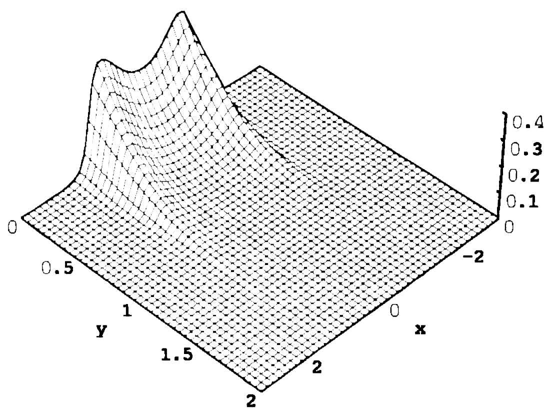


Fig. 5. Continued

the squeezed field of a damped harmonic oscillator with a squeezed bath. We gave the steady-state solution of the differential equation (Fokker–Planck) of the complex P-representation, which gives in one complex plane (i.e., $\alpha = \beta^*$) the Glauber–Sudarshan P-representation (Glauber, 1963). We have shown how the Pegg–Barnett phase distribution can be related to the phase produced by integrating the nondiagonal positive P-function. We also have shown the effect of the squeeze parameter contained in the basis state on the resulting phase distribution. This work should to be useful for the calculation of the quasiprobabilities for squeezed coherent states.

APPENDIX

We evaluate the integral

$$I = \int f(\alpha) \exp[-\nu_1|\alpha|^2 + \nu_2\alpha^2 + \nu_3\alpha^{*2} + \nu_4\alpha + \nu_5\alpha^*]\, d^2\alpha \quad \text{(A1)}$$

where ν_i, $i = 1, 2, \ldots, 5$ are complex numbers. The integral (Kral, 1990)

$$\int \exp[-\nu_1|\alpha|^2 + \nu_2\alpha^2 + \nu_3\alpha^{*2} + \nu_4\alpha + \nu_5\alpha^*]\, d^2\alpha$$

has the value

$$\frac{\pi}{\sqrt{K}} \exp\left[\frac{1}{K}(\nu_5\, \nu_4\nu_1 + \nu_5^2\nu_2 + \nu_4^2\nu_3)\right] \quad \text{(A2)}$$

where

$$Re\,(K) > 0, \qquad Re[\nu_1 + \nu_2 + \nu_3] > 0, \qquad K = \nu_1^2 - 4\nu_2\nu_3$$

Let the function $f(\alpha)$ be analytic and have a convergent series in the form

$$f(\alpha) = \sum_{m=0}^{\infty} C_m\alpha^m \quad \text{(A3)}$$

It is convenient to introduce the generating function

$$G(\eta) = \sum_{n=0}^{\infty} I_n \frac{\eta^n}{n!} \quad \text{(A4)}$$

where

$$I_n = \int \alpha^n\, exp[-\nu_1|\alpha|^2 + \nu_2\alpha^2 + \nu_3\alpha^{*2} + \nu_4\alpha + \nu_5\alpha^*]\, d^2\alpha \quad \text{(A5)}$$

By using (A2), we get

$$G(\eta) = \frac{\pi}{\sqrt{K}} \exp\left\{\frac{1}{K}[\nu_5\,(\nu_4 + \eta)\nu_1 + \nu_5^2\nu_2 + (\nu_4 + \eta)^2\nu_3]\right\} \quad \text{(A6)}$$

whence

$$I_n = \left(\frac{\partial}{\partial\eta}\right)^n G(\eta)|_{\eta=0} \tag{A7}$$

Consequently

$$I_n = \frac{\pi}{\sqrt{K}} \exp\left[\frac{1}{K}(\nu_5\nu_4\nu_1 + \nu_5^2\nu_2 + \nu_4^2\nu_3)\right]\left(\frac{\partial}{\partial\eta}\right)^n \times \left\{\exp\left[\frac{\eta}{K}(\nu_5\nu_1 + 2\nu_4\nu_3)\exp\left[\frac{\eta^2}{K}\nu_3\right]\right]\right\}\Bigg|_{\eta=0} \tag{A8}$$

By using the Leibniz formula

$$(Y\,Z)^{(n)} = \sum_{j=0}^{n}\binom{n}{j} Y^{(n-j)}Z^{(j)} \tag{A9}$$

and the above results, we get

$$\begin{aligned} I &= \int f(\alpha)\exp[-\nu_1|\alpha|^2 + \nu_2\alpha^2 + \nu_3\alpha^{*2} + \nu_4\alpha + \nu_5\alpha^*]\, d^2\alpha \\ &= \frac{\pi}{\sqrt{K}}\exp\left[\frac{1}{K}(\nu_5\nu_4\nu_1 + \nu_5^2\nu^2 + \nu_4^2\nu_3)\right]\left\{\sum_{n=0}^{\infty} C_n \sum_{s=0}^{n/2}\binom{n}{2s}\right. \\ &\times \left.\left(\frac{\nu_5\nu_1 + 2\nu_4\nu_3}{K}\right)^{n-2s}\left(\frac{\nu_3}{K}\right)^s \prod_{i=1}^{s}(2s - i + 1)\right\} \end{aligned} \tag{A10}$$

By using the generating function for the Hermite polynomials (Leonhardt and Jex, 1994), we may write (A10) in the form

$$I = \frac{\pi}{\sqrt{K}}\exp\left[\frac{1}{K}(\nu_5\nu_4\nu_1 + \nu_5^2\nu_2 + \nu_4^2\nu_3)\right] \times \left\{\sum_{n=0}^{\infty} C_n\left(-\frac{\nu_3}{K}\right)^{n/2} H_n(x)\right\} \tag{A11}$$

where

$$x = \left(\frac{\nu_5\nu_1 + 2\nu_4\nu_3}{2\sqrt{-K\nu_3}}\right)$$

and

$$H(x) = \frac{d^n}{dt^n} \exp[2xt - t^2]|_{t=0} \tag{A12}$$

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